

11005. Network Models and Graphical Models: A Survey

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Scope

- Network models represent dependencies with graphs; graphical models (one type of which are “Bayesian networks”) represent dependencies with graphs. That causes confusion. But they are very different!
 - Graphical models haven’t done the best job at networks, and networks haven’t made the best use of graphical models
- This talk is primarily to clarify (or create!) a *conceptual* connection between these two types of models
- For examples of *applications* of graphical modeling to networks, see Farasat et al. (2015), Maier et al. (2014), and Airolidi et al. (2008)

Key points

- **Graphical models** represent *dependencies* (and causal relationships) *between variables*
- **Networks models** are models of dyads, which represent *dependencies between observations*
 - Dyads can be modeled as random variables (e.g., Bernoulli for unweighted; Poisson for count; etc.)
 - Dyads are themselves dependent! (reciprocity, triadic closure, degree constraints)
 - Graphical models can represent these “dyadic dependencies”!

Key visualization

Terms: Snijders et al. (2006). From (old) joint work with Antonis Manousis and Naji Shajarisaies.

Factor graph	Parameter name	Network Motif	Parameterization	Matrix notation
	-mutual dyads		$\sum_{i < j} A_{ij} A_{ji}$	$\frac{1}{2} \text{tr}(\mathbf{AA}^T)$
	-in-two-stars		$\sum_{(i,j,k)} A_{ji} A_{ki}$	$\text{sum}(\mathbf{AA}^T) - \text{tr}(\mathbf{AA}^T)$
	-out-two-stars		$\sum_{(i,j,k)} A_{ij} A_{ik}$	$\text{sum}(\mathbf{A}^T \mathbf{A}) - \text{tr}(\mathbf{A}^T \mathbf{A})$
	-geom. weighted out-degrees	—	$\sum_i \exp\{-\alpha \sum_k A_{ik}\}$	$\text{sum}(\exp\{-\alpha \text{rowsum}(\mathbf{A})\})$
	-geom. weighted in-degrees	—	$\sum_j \exp\{-\alpha \sum_k A_{kj}\}$	$\text{sum}(\exp\{-\alpha \text{colsum}(\mathbf{A})\})$
	-alternating transitive k-triplets		$\lambda \sum_{i,j} A_{ij} \left\{ 1 - \left(1 - \frac{1}{\lambda}\right)^{\sum_{k \neq i,j} A_{ik} A_{kj}} \right\}$	$\lambda \text{sum}(\mathbf{A} \odot (1 - (1 - \frac{1}{\lambda})^{\mathbf{AA} - \text{diag}(\mathbf{AA})}))$
	-alternating indep. two-paths		$\lambda \sum_{i,j} \left\{ 1 - \left(1 - \frac{1}{\lambda}\right)^{\sum_{k \neq i,j} A_{ik} A_{kj}} \right\}$	$\lambda \text{sum}(1 - (1 - \frac{1}{\lambda})^{\mathbf{AA} - \text{diag}(\mathbf{AA})})$
	-two-paths (mixed two-stars)		$\sum_{(i,k,j)} A_{ik} A_{kj}$	$\text{sum}(\mathbf{AA}) - \text{tr}(\mathbf{AA})$
	-transitive triads		$\sum_{(i,j,k)} A_{ij} A_{jk} A_{ik}$	$\text{tr}(\mathbf{AAA}^T)$
	-activity effect		$\sum_i X_i \sum_j A_{ij}$	$\text{sum}(\mathbf{X} \odot \text{rowsum}(\mathbf{A}))$
	-popularity effect		$\sum_j X_j \sum_i A_{ij}$	$\text{sum}(\mathbf{X} \odot \text{colsum}(\mathbf{A}))$
	-similarity effect		$\sum_{i,j} A_{ij} \left(1 - \frac{ X_i - X_j }{\max_{k,l} X_k - X_l } \right)$	$\text{sum}(\mathbf{A} \odot \mathbf{S})$

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What are
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models?

Parameterizing
network models

Graphical
models for
networks

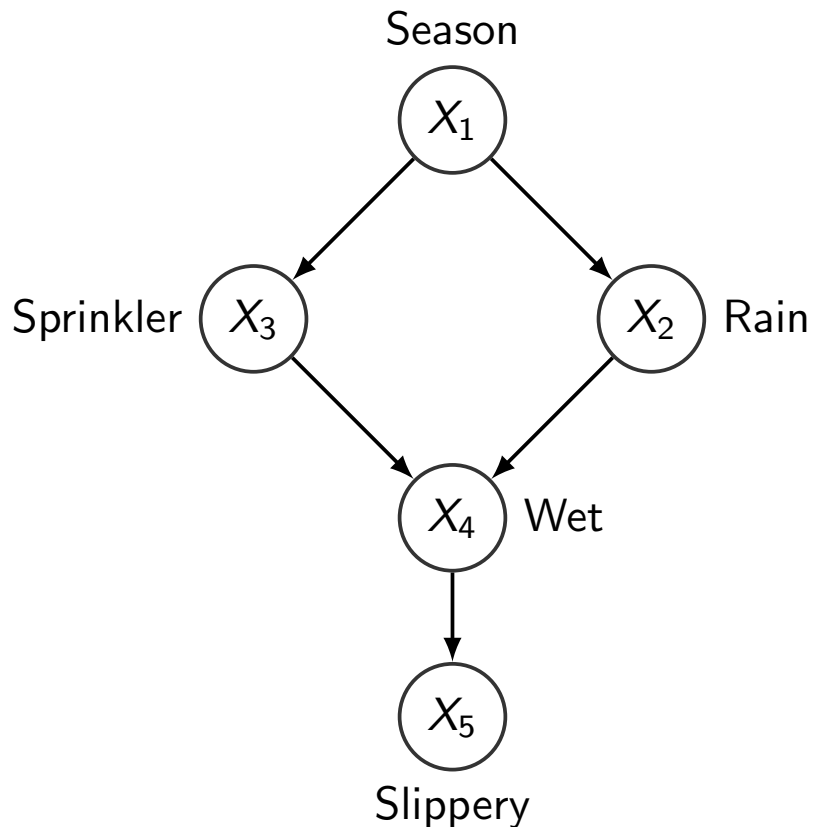
Conclusions

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What are graphical models?

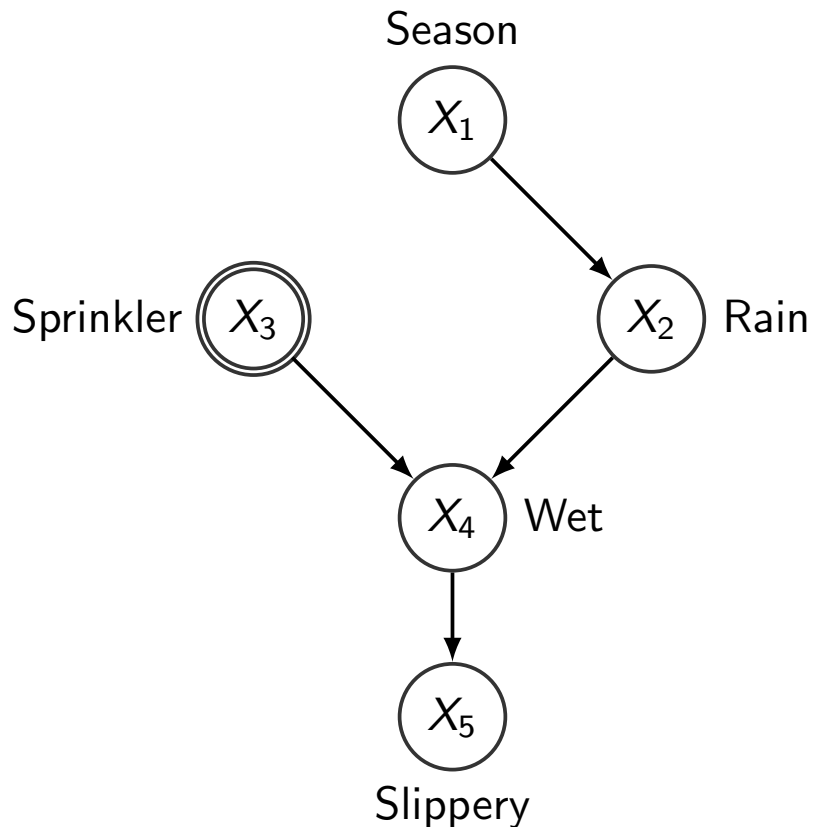
Graphical models = graphs for variables

- Like path diagrams in psych, but more formal. From CS in 90s
- Represent relationships between variables; can reason through dependencies
 - Sprinklers are not directly dependent on Rain, but if we know the grass is wet, we know *either* it rained *or* sprinklers were on (at least one is true)
- With probability distributions on the nodes, they represent *conditional independencies*
 - Equivalent to structural equation modeling (SEMs)!



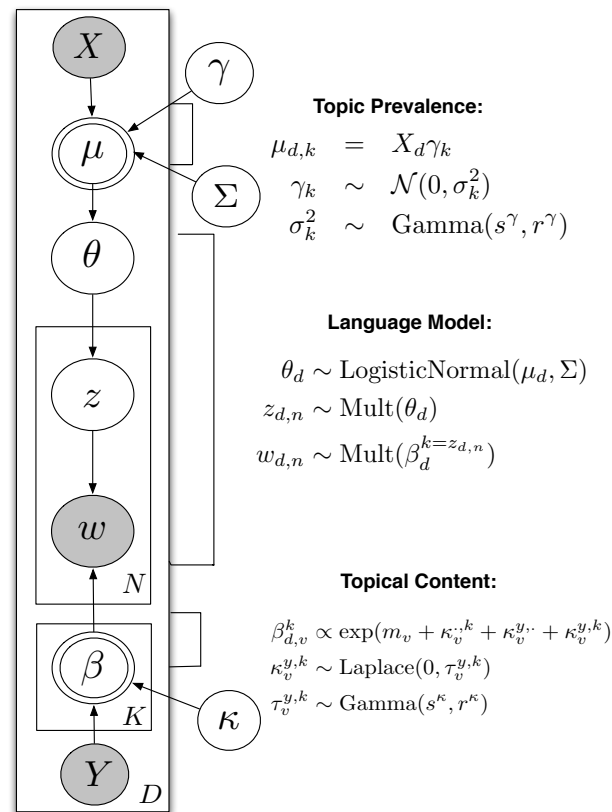
Can represent causality

- Can also reason about *causality*
- Interventions block “paths”
 - Pearl introduced the “do” operator to notate this algebraically
- Are algorithms to determine identifiability of parameters from a given (or assumed) causal structure (Bayes ball)
 - Causal *inference* techniques to estimate a causal graph (e.g., TETRAD algorithm) also exist, but theoretical guarantees for these procedures require strong, untestable, and almost certainly false assumptions



Mostly used for bookkeeping

- Most machine learning applications are effectively “bookkeeping”
- E.g., in structural topic modeling (Roberts et al., 2013), a topic doesn’t “cause” a document, but representing it as a directed tie is to help keep track of things for estimation



Networks in graphical models

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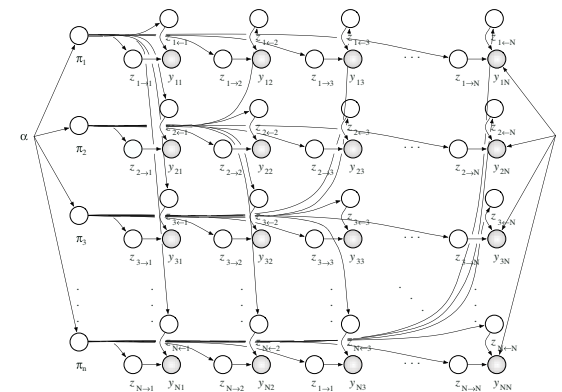
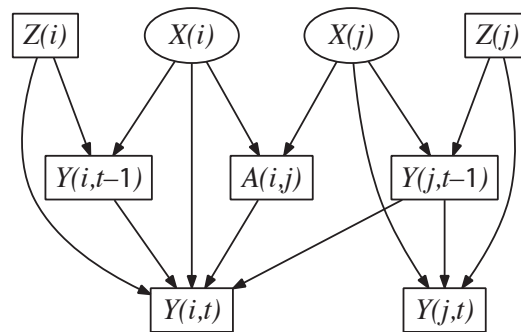
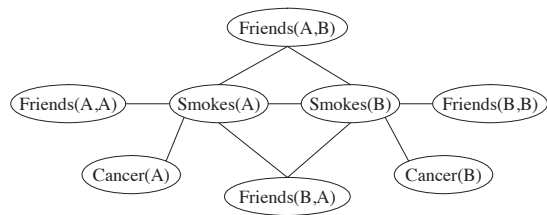
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Network edges as graphical model nodes in a “relational Markov network” (Getoor & Taskar, 2007): awkward

Network is only a confounder (Shalizi & Thomas, 2011), no direct dependencies between edges

Full representation, but only of a single type of dependency (block membership) (Airoldi et al., 2008)

Two “directions” of dependencies

- If covariates are independent (trivial graphical model, what we usually assume in linear regression), the joint distribution of a response and a design matrix is:

$$p(Y, \mathbf{X}) = p(Y, X_1, \dots, X_d) = p(Y|X_1, \dots, X_d) \prod_{j=1}^d p(X_j)$$

- (If we assume fixed \mathbf{X} , as we usually do, the probabilities of X_j 's go away)
- But a true, complete joint factorization of the conditional distribution would be over *observations* as well:

$$p(Y|\mathbf{X}) = p(y_1, \dots, y_n | x_{11}, \dots, x_{1d}, x_{21}, \dots, x_{2d}, \dots, x_{n1}, \dots, x_{nd})$$

- An iid assumption applies to the *observations*, and are how we even have multiple observations to estimate anything. This looks like:

$$p(Y|\mathbf{X}) = p(y_1, \dots, y_n | \mathbf{x}_1, \dots, \mathbf{x}_n) \stackrel{\text{iid}}{=} \prod_{i=1}^n p(y_i | \mathbf{x}_i)$$

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Networks: Dependencies as observations

Better way of looking at networks: make dyads the observations

- The response is now an edge, or edge attribute
- Transform all node covariates into edge covariates, e.g.,
 - As a difference between continuous node attributes
 - Indicator for if nodes in same category or not (or, make new categories out of possible pairs, e.g., $M \rightarrow M$, $M \rightarrow F$, $F \rightarrow M$, $F \rightarrow F$)
 - As “sender” and/or “receiver” attributes

	Y	X_1	X_2	\dots	X_d
1	y_1	x_{11}	x_{12}	\dots	x_{1d}
2	y_2	x_{21}	x_{22}	\dots	x_{2d}
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots
n	y_n	x_{n1}	x_{n2}	\dots	x_{nd}



$index$	$from$	to	Y	W_1	W_2	W_3	\dots
e_1	1	2	y_{12}	$\mathbf{1}(x_{11} = x_{21})$	$x_{12} - x_{22}$	x_{13}	\dots
e_2	2	3	y_{23}	$\mathbf{1}(x_{11} = x_{31})$	$x_{12} - x_{32}$	x_{13}	\dots
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
e_{n+1}	2	1	y_{21}	$\mathbf{1}(x_{21} = x_{11})$	$x_{22} - x_{12}$	x_{23}	\dots
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
$e_{2\binom{n}{2}}$	$n - 1$	n	$y_{(n-1)n}$	$\mathbf{1}(x_{(n-1)1} = x_{n1})$	$x_{(n-1)2} - x_{n2}$	$x_{(n-1)3}$	\dots

Parameterizing

- Descriptively,

$$a_{ij} = \begin{cases} 1 & \text{if there is a tie } i \rightarrow j \\ 0 & \text{otherwise.} \end{cases}$$

- Turn this into a random variable:

$$A_{ij} \stackrel{\text{iid}}{\sim} \text{Bernoulli}(p)$$

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As a logistic regression

- Add some covariates and it can become a logistic regression:

$$A_{ij} | x_i, x_j \sim \text{Bernoulli} \left(f(x_i, x_j)^T \beta \right)$$

$$\mathcal{L}(\beta) = \prod_{i=1}^n \prod_{j \neq i} f(x_i, x_j)^T \beta^{a_{ij}} (f(x_i, x_j)^T \beta)^{1-a_{ij}}$$

- The MLE of an intercept-only model is just the density.

$$\hat{\beta}_{MLE} = \frac{1}{2 \times \binom{n}{2}} \sum_{i \neq j} a_{ij}$$

Problem: Dyads are dependent, too

- In the language of ERGMs, “dyadic dependencies”
 - Social networks: reciprocity makes $A_{ij} \not\perp A_{ji}$
- The p_1 model (Holland & Leinhardt, 1981) deals with reciprocity as a one-off dependency by modeling edges as multinomial, with a cross term:

$$\mathbb{P}(A_{ij} = a_{ij}, A_{ji} = a_{ji}) = \frac{1}{k_{ij}} \exp \left\{ a_{ij}(\mu + \alpha_i + \beta_i) \right. \\ \left. + a_{ji}(\mu + \alpha_j + \beta_j) \right. \\ \left. + \rho a_{ij} a_{ji} \right\}$$

Models for dyadic dependencies

- Stochastic blockmodels (Wang & Wong, 1987): alternative to p_1 , two-level hierarchical version of Bernoulli model
- Latent space models (Hoff et al., 2002): can be seen as graphical models with observable nodes for edges, produced from hidden nodes representing latent position

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Markov property for network edges

- Landmark work: Frank & Strauss (1986)
- Markov dependence assumption: “A graph is said to be a Markov graph if only incident dyads can be conditionally dependent.”
- In retrospect, we can clarify this in terms of graphical models
 - The “graph” is the network, and the Markov property is of the graphical model of the network edges as Bernoulli variables

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Markov property for network edges

- Remarkably, using Hammersley-Clifford, Frank & Strauss proved that the graphical model of an undirected network is Markov if and only if

$$P_{\theta}(\mathbf{A}) = \frac{1}{\kappa(\theta)} \exp \left\{ \theta_0 L(\mathbf{A}) + \sum_{k=1}^{n-1} \theta_k S_k(\mathbf{A}) + \theta_{\tau} T(\mathbf{A}) \right\}$$

Normalization constant: need to sum over $2^{2 \times \binom{n}{2}}$ possible networks for each candidate θ

Non-maximal k -stars

Number of triangles

- Especially surprising part (Kolaczyk, 2009): how did triangles come out of this as a sufficient statistic??

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General modeling of dyadic dependencies

- This eventually led to Exponential-family Random Graph Models, which can model generic dependencies between edges
- Can add any sufficient statistic, although they can be collinear. E.g., two-paths are collinear with in-degrees, out-degrees, and mutual dyads (Snijders et al., 2006)

$$\sum_{i,j,k:k \neq i} A_{ij}A_{jk} = \sum_{j=1}^n \sum_{i,k:k \neq i} A_{ij}A_{jk} = \sum_{j=1}^n \left(A_{+j}A_{j+} - \sum_{i=1}^n A_{ij}A_{ji} \right)$$

- But: bad theoretical properties, tricky to estimate, and tricky to specify

Dyadic dependencies in a graph

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Key points, redux

- **Graphical models** represent *dependencies* (and causal relationships) *between variables*
- **Networks models** are models of dyads, which represent *dependencies between observations*
- They are not the same thing, but we can represent dyadic dependencies (dependencies between edges of a network, in processes like reciprocity and transitivity) as graphical models

Why should we care?

- Graphical models and network models are both powerful for representing, reasoning through, and modeling dependencies
 - But graphical models haven't done the best job at networks, and networks haven't made the best use of graphical models
 - Maybe because they are used by different communities, and the same words (“networks”, “dependencies”) mean subtly different things
- Clarifying the relationship of these two types of models helps head off confusion, as well as deepen our appreciation of the idea of “dependencies”
 - As well as helping to train students
- Practically: Can graphical models help create, and estimate, new statistical network models, and unify existing ones? Almost certainly, although in many cases the estimation will still be MCMC (maybe variational inference; Celisse et al., 2012)

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